THE NUMBER e AND THE NATURAL LOGARITHM
COMMON CORE ALGEBRA II

There are many numbers in mathematics that are more important than others because they find so many uses in either mathematics or science. Good examples of important numbers are 0, 1, \( i \), and \( \pi \). In this lesson you will be introduced to an important number given the letter \( e \) for its “inventor” Leonhard Euler (1707-1783). This number plays a crucial role in Calculus and more generally in modeling exponential phenomena.

### The Number \( e \)

1. Like \( \pi \), \( e \) is irrational.
2. \( e \approx 2.72 \)
3. Used in Exponential Modeling

**Exercise #1:** Which of the graphs below shows \( y = e^x \)? Explain your choice. Check on your calculator.

(1) \[ y = e^x \] (2) \[ y = e^x \] (3) \[ y = e^x \] (4) \[ y = e^x \]

**Explanation:**

Since \( e \) is a number greater than one, the graph of \( y = e^x \) must be that of an increasing exponential function.

Very often \( e \) is involved in exponential modeling of both increasing and decreasing quantities. The creation of these models is beyond the scope of this course, but we can still work with them.

**Exercise #2:** A population of llamas on a tropical island can be modeled by the equation \( P = 500e^{0.035t} \), where \( t \) represents the number of years since the llamas were first introduced to the island.

(a) How many llamas were initially introduced at \( t = 0 \)? Show the calculation that leads to your answer.

\[
P = 500e^{0.035 \cdot 0} \\
= 500e^0 \\
= 500 \cdot 1 \\
= 500
\]

(b) Algebraically determine the number of years for the population to reach 600. Round your answer to the nearest tenth of a year.

\[
500e^{0.035t} = 600 \\
e^{0.035t} = \frac{600}{500} = 1.2 \\
\log(e^{0.035t}) = \log 1.2 \\
0.035t \cdot \log e = \log 1.2 \\
t = \frac{\log 1.2}{0.035 \log e} \approx 5.2
\]
Because of the importance of $y = e^x$, its inverse, known as the **natural logarithm**, is also important.

### The Natural Logarithm

The inverse of $y = e^x$: $y = \ln x \quad (y = \log_e x)$

The natural logarithm, like all logarithms, gives an exponent as its output. In fact, it gives the power that we must raise $e$ to in order to get the input.

**Exercise #3:** Without the use of your calculator, determine the values of each of the following.

(a) $\ln e$

(b) $\ln 1$

(c) $\ln e^5$

(d) $\ln \sqrt{e}$

The natural logarithm follows the three basic logarithm laws that all logarithms follow. The following problems give additional practice with these laws.

**Exercise #4:** Which of the following is equivalent to $\ln \left( \frac{x^3}{e^2} \right)$?

(1) $\ln x + 6$

(2) $3 \ln x - 6$

(3) $3 \ln x - 2$

(4) $\ln x - 9$

**Exercise #5:** A hot liquid is cooling in a room whose temperature is constant. Its temperature can be modeled using the exponential function shown below. The temperature, $T$, is in degrees Fahrenheit and is a function of the number of minutes, $m$, it has been cooling.

$$T(m) = 101e^{-0.03m} + 67$$

(a) What was the initial temperature of the water at $m = 0$. Do without using your calculator.

$$T(0) = 101e^{-0.03\cdot0} + 67 = 101e^0 + 67$$

$$= 101(1) + 67 = 101 + 67 = 168$$

(c) Using the natural logarithm, determine algebraically when the temperature of the liquid will reach 100 °F. Show the steps in your solution. Round to the nearest tenth of a minute.

(d) On average, how many degrees are lost per minute over the interval $10 \leq m \leq 30$? Round to the nearest tenth of a degree.

The temperature of the liquid after 60 minutes or one hour is 83.7 degrees Fahrenheit.
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FLUENCY

1. Which of the following is closest to the y-intercept of the function whose equation is \( y = 10e^{x+1} \)?

   (1) 10  (3) 27
   (2) 18  (4) 52

\[ y = 10e^{x+1} = 10e^1 \approx 27 \]

2. On the grid below, the solid curve represents \( y = e^x \). Which of the following exponential functions could describe the dashed curve? Explain your choice.

   (1) \( y = \left(\frac{1}{2}\right)^x \)
   (2) \( y = e^{-x} \)
   (3) \( y = 2^x \)
   (4) \( y = 4^x \)

The graph in dashed rises faster than \( y = e^x \), so it must have a larger base.

3. The logarithmic expression \( \ln\left(\frac{\sqrt{e}}{y^3}\right) \) can be rewritten as

   (1) \( 3\ln y - 2 \)
   (2) \( \frac{1 - 6\ln y}{2} \)
   (3) \( \frac{\ln y - 6}{2} \)
   (4) \( \sqrt{\ln y} - 3 \)

\[ \frac{1}{2} \ln y = \frac{1 - 6\ln y}{2} \]

4. Which of the following values of \( t \) solves the equation \( 5e^{2t} = 15 \)?

   (1) \( \frac{\ln 15}{10} \)
   (2) \( \frac{1}{2\ln 5} \)
   (3) \( 2\ln 3 \)
   (4) \( \frac{\ln 3}{2} \)

\( 5e^{2t} = 15 \)
\[ e^{2t} = 3 \]
\[ \ln(e^{2t}) = \ln 3 \]
\[ 2t = \ln 3 \]
\[ t = \frac{\ln 3}{2} \]

5. At which of the following values of \( x \) does \( f(x) = 2e^{2x} - 32 \) have a zero?

   (1) \( \ln \frac{5}{2} \)
   (2) \( \ln 4 \)
   (3) \( \ln 8 \)
   (4) \( y = \ln \frac{2}{5} \)

\( 2e^{2x} - 32 = 0 \)
\[ 2e^{2x} = 32 \]
\[ e^{2x} = 16 \]
\[ \ln(e^{2x}) = \ln 16 \]
\[ 2x = \ln 16 \]
\[ x = \frac{1}{2} \ln 16 \]
\[ x = \ln 16^{\frac{1}{2}} \]
\[ x = \ln 4 \]
6. For the equation \(ae^{ct} = d\), solve for the variable \(t\) in terms of \(a, c,\) and \(d\). Express your answer in terms of the natural logarithm.

\[
\begin{align*}
ae^{ct} &= d \\
e^{ct} &= \frac{d}{a} \\
\ln(e^{ct}) &= \ln\left(\frac{d}{a}\right) \\
t &= \frac{\ln\left(\frac{d}{a}\right)}{c}
\end{align*}
\]

APPLICATIONS

7. Flu is spreading exponentially at a school. The number of new flu patients can be modeled using the equation \(F = 10e^{0.12d}\), where \(d\) represents the number of days since 10 students had the flu.

(a) How many days will it take for the number of new flu patients to equal 50? Determine your answer algebraically using the natural logarithm. Round your answer to the nearest day.

\[
\begin{align*}
10e^{0.12d} &= 50 \\
e^{0.12d} &= 5 \\
\ln(e^{0.12d}) &= \ln 5 \\
0.12d &= \ln 5 \\
d &= \frac{\ln 5}{0.12} \\
&\approx 13 \text{ days}
\end{align*}
\]

(b) Find the average rate of change of \(F\) over the first three weeks, i.e. \(0 \leq d \leq 21\). Show the calculation that leads to your answer. Give proper units and round your answer to the nearest tenth. What is the physical interpretation of your answer?

\[
\begin{align*}
\text{On average, the number of new flu patients was increasing at a rate of 5.4 flu patients per day.}
\end{align*}
\]

8. The savings in a bank account can be modeled using \(S = 1250e^{0.045t}\), where \(t\) is the number of years the money has been in the account. Determine, to the nearest tenth of a year, how long it will take for the amount of savings to double from the initial amount deposited of $1250.

\[
\begin{align*}
1250e^{0.045t} &= 2500 \\
e^{0.045t} &= 2 \\
\ln(e^{0.045t}) &= \ln 2 \\
0.045t &= \ln 2 \\
0.045t &= 0.693147... \\
t &= \frac{\ln 2}{0.045} \\
&\approx 15.4 \text{ yrs}
\end{align*}
\]