**SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS**

**COMMON CORE ALGEBRA II**

Earlier in this unit, we used the **Method of Common Bases** to solve exponential equations. This technique is quite limited, however, because it requires the two sides of the equation to be expressed using the same base. A more general method utilizes our calculators and the third logarithm law:

### The Third Logarithm Law

\[
\log_b \left( a^x \right) = x \log_b a
\]

**Exercise #1:** Solve: \(4^x = 8\) using (a) common bases and (b) the logarithm law shown above.

(a) Method of Common Bases

\[
\left(2^2\right)^x = 2^3 \Rightarrow 2^{2x} = 2^3 \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2} = 1.5
\]

(b) Logarithm Approach

\[
\log(4^x) = \log(8) \\
x \cdot \log 4 = \log 8 \\
x = \frac{\log 8}{\log 4} \approx 1.5
\]

The beauty of this logarithm law is that it removes the variable from the exponent. This law, in combination with the logarithm base 10, the **common log**, allows us to solve almost any exponential equation.

**Exercise #2:** Solve each of the following equations for the value of \(x\). Round your answers to the nearest hundredth.

(a) \(5^x = 18\) 

\[
\log(5^x) = \log 18 \\
x \cdot \log 5 = \log 18 \Rightarrow x = \frac{\log 18}{\log 5} \approx 1.80
\]

(b) \(4^x = 100\) 

\[
\log(4^x) = \log 100 \\
x \cdot \log 4 = \log 100 \Rightarrow x = \frac{\log 100}{\log 4} \approx 3.32
\]

(c) \(2^x = 1560\) 

\[
\log(2^x) = \log 1560 \\
x \cdot \log 2 = \log 1560 \Rightarrow x = \frac{\log 1560}{\log 2} \approx 10.61
\]

These equations can become more complicated, but each and every time we will use the logarithm law to transform an exponential equation into one that is more familiar (linear, quadratic, etc).

**Exercise #3:** Solve each of the following equations for \(x\). Round your answers to the nearest hundredth.

(a) \(6^{x+3} = 50\)

\[
\log(6^{x+3}) = \log 50 \\
(x + 3) \log 6 = \log 50 \Rightarrow x + 3 = \frac{\log 50}{\log 6} \\
x = \frac{\log 50}{\log 6} - 3 \approx -0.82
\]

(b) \((1.03)^{\frac{x}{2}} = 2\)

\[
\log \left(1.03^{\frac{x}{2}}\right) = \log 2 \Rightarrow \left(\frac{x}{2} - 5\right) \log 1.03 = \log 2 \\
x = \frac{2}{\log 1.03} \Rightarrow x = \frac{\log 2}{\log 1.03} + 5 \\
x = 2 \left(\frac{\log 2}{\log 1.03} + 5\right) \approx 56.90
\]
Now that we are familiar with this method, we can revisit some of our exponential models from earlier in the unit. Recall that for an exponential function that is growing:

If quantity $Q$ is known to increase by a fixed percentage $p$, in decimal form, then $Q$ can be modeled by

$$Q(t) = Q_0 \left(1 + \frac{p}{100}\right)^t$$

where $Q_0$ represents the amount of $Q$ present at $t = 0$ and $t$ represents time.

**Exercise #4:** A biologist is modeling the population of bats on a tropical island. When he first starts observing them, there are 104 bats. The biologist believes that the bat population is growing at a rate of 3% per year.

(a) Write an equation for the number of bats, $B(t)$, as a function of the number of years, $t$, since the biologist started observing them.

$$B(t) = 104 \left(1 + \frac{0.03}{100}\right)^t$$

(b) Using your equation from part (a), algebraically determine the number of years it will take for the bat population to reach 200. Round your answer to the nearest year.

$$104 \left(1.03\right)^t = 200$$

$$t \log 1.03 = \log \left(\frac{200}{104}\right) \Rightarrow t \approx 22$$

**Exercise #5:** A stock has been declining in price at a steady pace of 5% per week. If the stock started at a price of $22.50 per share, determine algebraically the number of weeks it will take for the price to reach $10.00. Round your answer to the nearest week.

$$22.50 \left(1 - \frac{0.05}{100}\right)^t = 10.00$$

$$0.95^t = \frac{10.00}{22.50} \Rightarrow t \log 0.95 = \log \left(\frac{10.00}{22.50}\right) \Rightarrow t \approx 16$$

As a final discussion, we return to evaluating logarithms using our calculator. Many modern calculators can find a logarithm of any base. Some still only have the common log (base 10) and another that we will soon see. But, we can still express our answers in terms of logarithms.

**Exercise #6:** Find the solution to each of the following exponential equations in terms of a logarithm with the same base as the exponential equation.

(a) $4 \left(2^x\right)^2 - 3 = 17$

$$4 \left(2^x\right)^2 = 20$$

$$2^x = 5$$

$$x = \log_2(5)$$

(b) $17 \left(5^{\frac{x}{5}}\right) = 4$

$$17 \cdot 5^{\frac{x}{5}} = 4$$

$$\frac{x}{5} = \log_5\left(\frac{4}{17}\right)$$

$$x = 3 \log_5\left(\frac{4}{17}\right)$$
SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following values, to the nearest hundredth, solves: $7^x = 500$?
   
   (1) 3.19  (3) 2.74
   (2) 3.83  (4) 2.17

   \[
   \log(7^x) = \log(500) \\
   x \log 7 = \log 500 \Rightarrow x = \frac{\log 500}{\log 7} \approx 3.19
   \]

2. The solution to $2^{\frac{x}{3}} = 52$, to the nearest tenth, is which of the following?
   
   (1) 7.3  (3) 11.4
   (2) 9.1  (4) 17.1

   \[
   \log \left(2^{\frac{x}{3}}\right) = \log 52 \Rightarrow \frac{x}{3} \log 2 = \log 52 \\
   x \approx 3 \left(\frac{\log 52}{\log 2}\right) \approx 17.1
   \]

3. To the nearest hundredth, the value of $x$ that solves $4^{x-4} = 275$ is
   
   (1) 6.73  (3) 8.17
   (2) 5.74  (4) 7.49

   \[
   \log \left(4^{x-4}\right) = \log 275 \Rightarrow (x - 4) \log 5 = \log 275 \\
   x - 4 = \frac{\log 275}{\log 5} \Rightarrow x = \frac{\log 275}{\log 5} + 4 \approx 7.49
   \]

4. Solve each of the following exponential equations. Round each of your answers to the nearest hundredth.

   (a) $9^{x-3} = 250$

   \[
   \log \left(9^{x-3}\right) = \log 250 \\
   (x - 3) \log 9 = \log 250 \Rightarrow x - 3 = \frac{\log 250}{\log 9} \\
   x = \frac{\log 250}{\log 9} + 3 \approx 5.51
   \]

   (b) $50(2)^x = 1000$

   \[
   2^x = 20 \Rightarrow \log \left(2^x\right) = \log 20 \\
   x \log 2 = \log 20 \Rightarrow x = \frac{\log 20}{\log 2} \approx 4.32
   \]

   (c) $5^{\frac{x}{10}} = 35$

   \[
   \log \left(5^{\frac{x}{10}}\right) = \log 35 \Rightarrow \frac{x}{10} \log 5 = \log 35 \\
   x = 10 \left(\frac{\log 35}{\log 5}\right) \approx 22.09
   \]

5. Solve each of the following exponential equations. Be careful with your use of parentheses. Express each answer to the nearest hundredth.

   (a) $6^{2x-5} = 300$

   \[
   \log \left(6^{2x-5}\right) = \log 300 \\
   (2x - 5) \log 6 = \log 300 \Rightarrow 2x = \frac{\log 300}{\log 6} + 5 \\
   x = \frac{\log 300}{\log 6} + 5 \approx 4.09
   \]

   (b) \(\left(\frac{1}{2}\right)^{x+1} = \frac{1}{6}\)

   \[
   \log \left(\frac{1}{2}\right)^{x+1} = \log \left(\frac{1}{6}\right) \\
   (x + 1) \log \left(\frac{1}{2}\right) = \log \left(\frac{1}{6}\right) \\
   x + 1 = \frac{\log \left(\frac{1}{6}\right)}{\log \left(\frac{1}{2}\right)} \Rightarrow x = 3 \left(\frac{\log \left(\frac{1}{6}\right)}{\log \left(\frac{1}{2}\right)} - 1\right) \\
   x \approx 4.75
   \]

   (c) $500(1.02)^\frac{x}{12} = 2300$

   \[
   \log \left(500(1.02)^{\frac{x}{12}}\right) = \log 2300 \\
   \log 1.02 = \frac{2300}{500} = 4.6 \\
   \log \left(1.02^{\frac{x}{12}}\right) = \log 4.6 \\
   \frac{x}{12} \log 1.02 = \log 4.6 \\
   x = 12 \left(\frac{\log 4.6}{\log 1.02}\right) \approx 924.76
   \]
APPLICATIONS

6. The population of Red Hook is growing at a rate of 3.5% per year. If its current population is 12,500, in how many years will the population exceed 20,000? Round your answer to the nearest year. Only an algebraic solution is acceptable.

\[
12500 \times (1.035)^t = 20000 \\
1.035^t = \frac{20000}{12500} \\
\log(1.035^t) = \log\left(\frac{20000}{12500}\right) \\
t \approx \frac{\log(20000/12500)}{\log(1.035)} \approx 14 \text{ yrs.}
\]

8. A radioactive substance is decaying such that 2% of its mass is lost every year. Originally there were 50 kilograms of the substance present.

(a) Write an equation for the amount, \(A\), of the substance left after \(t\)-years.

\[
A = 50(1 - 0.02)^t \\
A = 50(0.98)^t
\]

(b) Find the amount of time that it takes for only half of the initial amount to remain. Round your answer to the nearest tenth of a year.

\[
50(0.98)^t = 25 \Rightarrow 0.98^t = \frac{25}{50} = 0.5
\]

\[
\log(0.98)^t = \log 0.5 \\
t \log 0.98 = \log 0.5 \\
t = \frac{\log 0.5}{\log 0.98} \approx 34.3 \text{ yrs}
\]

REASONING

8. If a population doubles every 5 years, how many years will it take for the population to increase by 10 times its original amount?

First: If the population gets multiplied by 2 every 5 years, what does it get multiplied by each year? Use this to help you answer the question.

Since the population doubles every 5 years, then we know that we multiply the population every year by:

\[
2^{\frac{x}{5}}
\]

\[
\left(2^{\frac{x}{5}}\right)^x = 10 \\
2^{\frac{x}{5}} = 10
\]

\[
\log \left(2^{\frac{x}{5}}\right) = \log(10)
\]

\[
\frac{x}{5} \log(2) = 1 \\
x = \frac{5}{\log(2)} = 16.6096... \\
So \approx 16.6 \text{ years.}
\]

9. Find the solution to the general exponential equation \(a(b)^x = d\), in terms of the constants \(a, c, d\) and the logarithm of base \(b\). Think about reversing the order of operations in order to solve for \(x\).

\[
\frac{a(b)^x}{a} = \frac{d}{a} \\
b^cx = \frac{d}{a}
\]

\[
x = \frac{1}{c} \log_b \left(\frac{d}{a}\right) \quad \text{or} \quad x = \frac{\log_b \left(\frac{d}{a}\right)}{c}
\]