UNIT #5

COMPLEX NUMBERS

Lesson #1 – Imaginary Numbers

Lesson #2 – Introduction to Complex Numbers

Lesson #3 – Multiplication and Division of Complex Numbers

Lesson #4 – Solving Quadratic Equations with Complex Solutions

Lesson #5 – The Discriminant of a Quadratic

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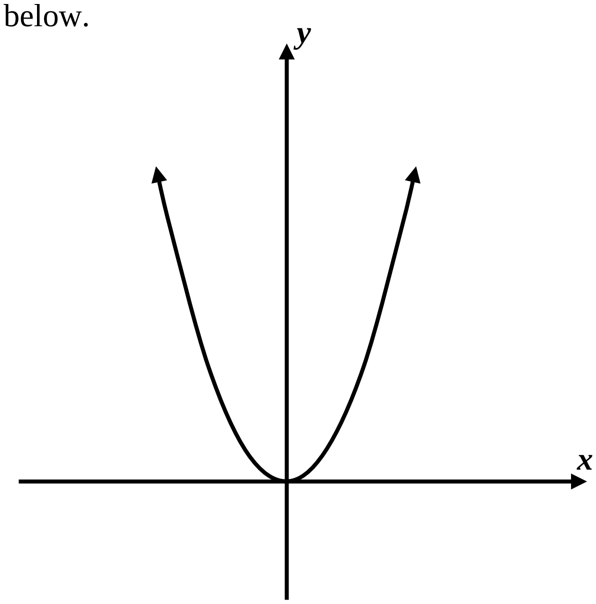
IMAGINARY NUMBERS
ALGEBRA 2 WITH TRIGONOMETRY

Recall that in the Real Number System, it is not possible to take the square root of a negative quantity because whenever a real number is squared it is non-negative. This fact has a ramification for finding the *x*-intercepts of a parabola, as *Exercise* #1 will illustrate.

***Exercise* #1:** On the axes below, a sketch of $y = x^2$ is shown. Now, consider the parabola whose equation is given in function notation as $f(x) = x^2 + 1$.

(a) How is the graph of $y = x^2$ shifted to produce the graph of $f(x)$?

(b) Create a quick sketch of $f(x)$ on the axes below.



(c) What can be said about the *x*-intercepts of the function $y = f(x)$?

(d) Algebraically, show that these intercepts do not exist, in the Real Number System, by solving the incomplete quadratic $x^2 + 1 = 0$.

Since we cannot solve this equation using Real Numbers, we introduce a new number, called *i*, the basis of imaginary numbers. Its definition allows us to now have a result when finding the square root of a negative real number. Its definition is given below.

THE DEFINITION OF THE IMAGINARY NUMBER *i*

$$i = \sqrt{-1}$$

***Exercise* #2:** Simplify each of the following square roots in terms of *i*.

(a) $\sqrt{-9}$ (b) $\sqrt{-100}$ (c) $\sqrt{-32}$ (d) $\sqrt{-18}$

***Exercise* #2:** Solve each of the following incomplete quadratics. Place your answers in simplest radical form.

(a) $5x^2+8=-12$ (b) $\frac{1}{2}x^2+20=2$ (c) $2x^2-10=-36$

***Exercise* #3:** Which of the following is equivalent to $5i \cdot 6i$?

(1) $30i$ (3) -30

(2) $11i$ (4) -11

Powers of i display a remarkable pattern that allow us to simplify large powers of i into one of 4 cases. This pattern is discovered in *Exercise* #4.

***Exercise* #4:** Simplify each of the following powers of i.

$i^1=i$ $i^2=$ $i^3=$ $i^4=$

$i^5=$ $i^6=$ $i^7=$ $i^8=$

We see, then, from this pattern that every power of i is either $-1, 1, i,$ or $-i$. And the pattern will repeat.

***Exercise* #5:** From the pattern of *Exercise* #4, simplify each of the following powers of i.

(a) $i^{38}=$ (b) $i^{21}=$ (c) $i^{83}=$ (d) $i^{40}=$

***Exercise* #6:** Which of the following is equivalent to $5i^{16}+3i^{23}+i^{26}$?

(1) $8+2i$ (3) $5-4i$

(2) $4-3i$ (4) $2+7i$

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IMAGINARY NUMBERS
ALGEBRA 2 WITH TRIGONOMETRY - HOMEWORK

SKILLS

1. The imaginary number i is defined as

(1) -1 (3) $\sqrt{-4}$

(2) $\sqrt{-1}$ (4) $(-1)^2$ ________

2. Which of the following is equivalent to $\sqrt{-128}$?

(1) $8\sqrt{2}$ (3) $-8\sqrt{2}$

(2) $8i$ (4) $8i\sqrt{2}$ ________

3. The sum $\sqrt{-9}+\sqrt{-16}$ is equal to

(1) 5 (3) $7i$

(2) $5i$ (4) 7 ________

4. Which of the following powers of i is *not* equal to one?

(1) i^{16} (3) i^{32}

(2) i^{26} (4) i^{48} ________

5. Which of the following represents all solutions to the equation $\frac{1}{3}x^2+10=7$?

(1) $x=\pm 3i$ (3) $x=\pm i$

(2) $x=\pm 5i$ (4) $x=\pm 2i$ ________

6. Solve each of the following incomplete quadratics. Express your answers in simplest radical form.

(a) $2x^2+100=-62$

(b) $\frac{2}{3}x^2+20=2$

7. Which of the following represents the solution set of $\frac{1}{2}x^2 - 12 = -37$?

(1) $\pm 7i$ (3) $\pm 5i\sqrt{2}$

(2) $\pm 7i\sqrt{2}$ (4) $\pm 3i\sqrt{2}$

8. Simplify each of the following powers of i into either $-1, 1, i,$ or $-i$.

(a) i^2 (b) i^3 (c) i^4 (d) i^{11}

(e) i^{41} (f) i^{30} (g) i^{25} (h) i^{36}

(i) i^{51} (j) i^{45} (k) i^{80} (l) i^{70}

9. Which of the following is equivalent to $i^7 + i^8 + i^9 + i^{10}$?

(1) 1 (3) $1 - i$

(2) $2 + i$ (4) 0

10. When simplified the sum $5i^{18} + 7i^{25} + 2i^{28} + 6i^{43}$ is equal to

(1) $2 - 4i$ (3) $5 - 7i$

(2) $-3 + i$ (4) $8 + i$

11. The product $(6 + 2i)(4 - 3i)$ can be written as

(1) $24 - 6i$ (3) $2 + 5i$

(2) $18 + 10i$ (4) $30 - 10i$

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INTRODUCTION TO COMPLEX NUMBERS
ALGEBRA 2 WITH TRIGONOMETRY

All numbers fall into a very broad category known as complex numbers. Complex numbers can always be thought of as a combination of a real number with an imaginary number and will have the form:

$$a + bi \text{ where } a \text{ and } b \text{ are real numbers}$$

We say that a is the real part of the number and bi is the imaginary part of the number.

Graphing Complex Numbers – Complex numbers can be graphed, just like real numbers. Unlike real numbers, since complex numbers have two components, real and imaginary, they must be plotted in a two-dimensional system. This two dimensional system is called the **complex plane** or the **Argand plane**.

***Exercise* #1:** Plot each of the following complex numbers on the complex plane shown below. Label them by letter.

(a) $5-2i$

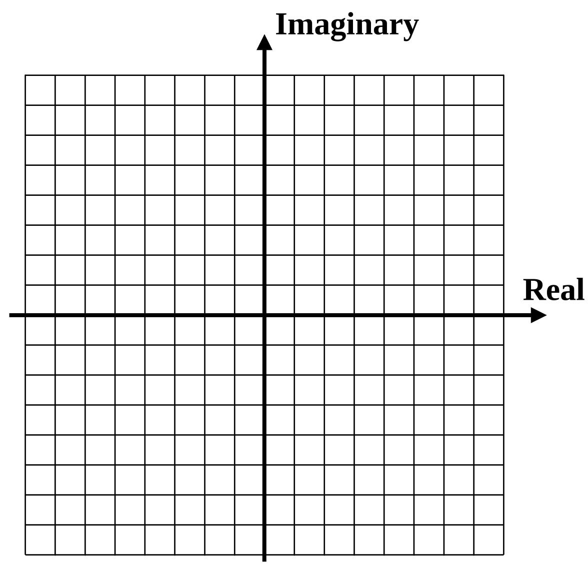
(b) $3+7i$

(c) $-2+6i$

(d) $-4-i$

(e) $5i$

(f) -6



***Exercise* #2:** If the complex number $-3+5i$ were plotted in the complex plane, it would land in which of the following quadrants?

(1) I (3) III

(2) II (4) IV ________

Finding the Size or Modulus of a Complex Number – Although it is not possible to say whether one complex number is greater than another, it is possible to compare their sizes. The size, or modulus, of a complex number is calculated as

$$|a+bi| = \sqrt{a^2 + b^2}$$

***Exercise* #3:** Find the size of each of the following complex numbers. Simplify your answers.

(a) $3-4i$ (b) $12+5i$ (c) $-4+2i$

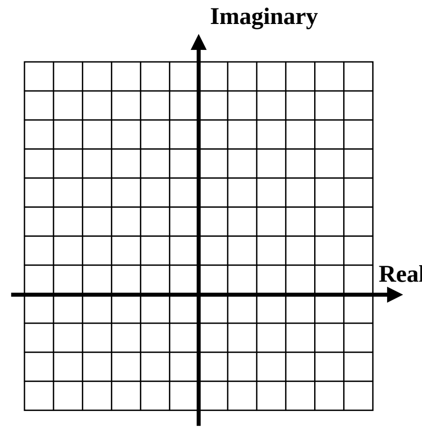
Notice that the modulus of a complex number is a real number.

***Exercise* #4:** Consider the complex number $-3+6i$.

(a) Plot this number on the plane below.

(b) Find the modulus of this number in simplest form.

(c) What distance does the modulus represent on the graph? Explain by illustrating your answer on the graph.



Like real numbers, complex numbers may be added and subtracted. The key to these operations is that real components can combine with real components and imaginary with imaginary.

***Exercise* #5:** Find each of the following sums and differences.

(a) $(-2+7i)+(6+2i)$ (b) $(8+4i)+(12-i)$ (c) $(5+3i)-(2-7i)$ (d) $(-3+5i)-(-8+2i)$

***Exercise* #6:** If the sum of $(6+2i)$ and $(-8-5i)$ were plotted in the complex plane, in which quadrant would it fall?

(1) I (3) III

(2) II (4) IV

____

***Exercise* #7:** When plotted, which of the following complex numbers would fall furthest from the origin?

(1) $3+4i$ (3) $-2+8i$

(2) $4-7i$ (4) $1+6i$

____

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INTRODUCTION TO COMPLEX NUMBERS
ALGEBRA 2 WITH TRIGONOMETRY - HOMEWORK

SKILLS

1. Plot each of the following complex numbers on the plane below. Label each with its letter.

(a) $2+5i$

(b) $-3-4i$

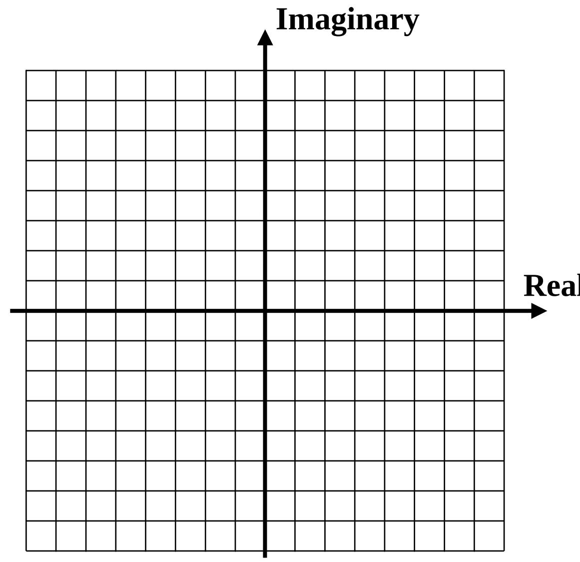
(c) $7-2i$

(d) 4

(e) $-6i$

(f) $-5+5i$

(g) $2-8i$



2. Find the modulus of each of the following complex numbers in simplest form.

(a) $-5+12i$ (b) $2-4i$ (c) $-6-8i$

(d) $10+5i$ (e) $-3+9i$ (f) $-4-8i$

3. When plotted in the complex plane, which of the following numbers is furthest from the origin?

(1) $7+4i$ (3) $-5-11i$

(2) $8-2i$ (4) $-9+8i$

____

4. Of the fours choices below, which is closest to the origin when plotted in the complex plane?

(1) $-6+10i$ (3) $7-2i$

(2) $-3-3i$ (4) $5+i$

______

5. Find each of the following sum or difference.

(a) $(6+3i)+(-2+9i)$ (b) $(-7+i)-(3+5i)$ (c) $(10-3i)+(6-8i)$

(d) $(-2+7i)-(15-6i)$ (e) $(15+2i)+(5-5i)$ (f) $(-1+i)-(-5-6i)$

6. When the numbers $(4-2i)$ and $2-5i$ are added and the result is plotted, it will fall in which quadrant?

(1) I (3) III

(2) II (4) IV

______

7. If $(3-5i)$ is subtracted from $(1-3i)$ the result would be plotted in which of the following quadrants of the complex plane?

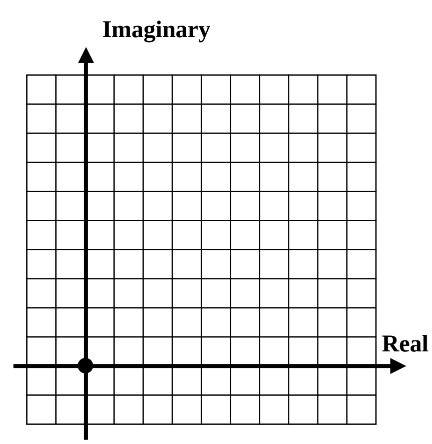
(1) I (3) III

(2) II (4) IV

______

REASONING

8. Consider the two complex numbers $(3+4i)$ and $(5+3i)$.



(a) Plot both numbers on the complex plane given.

(b) Find the sum of these numbers and plot it as well.

(c) Along with the origin, connect these four points to form a quadrilateral.

(d) What special type of quadrilateral is this?

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MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS
ALGEBRA 2 WITH TRIGONOMETRY

The set of complex numbers is **closed** under the operations of multiplication and division (except by zero). What this means is that when two complex numbers are multiplied, their product is another complex number - similarly for division. Multiplication is easy enough because it follows the basic principle of multiplying two binomials.

***Exercise* #1:** Find the following products. Write each of your answers as a complex number in the form $a+bi$.

(a) $(3+5i)(7+2i)$ (b) $(-2+6i)(3-2i)$ (c) $(4+i)(-5-3i)$

A curious result occurs when multiplying two numbers known as **complex conjugates**. The numbers $(a+bi)$ and $(a-bi)$ are the general form of complex conjugates.

***Exercise* #2:** Find each of the following products of complex conjugates.

(a) $(4+2i)(4-2i)$ (b) $(-5+i)(-5-i)$ (c) $(6+3i)(6-3i)$

***Exercise* #3:** What is true about each of the products that you found in *Exercise* #2?

***Exercise* #4:** Show that the product of a complex number $a+bi$ with its conjugate is always a purely real number and that number is equal to a^2+b^2.

Dividing complex numbers is not as straightforward as multiplying them. It is not at all obvious why two numbers of the form $a+bi$ should divide to be another of this form. The key is in the conjugate multiplication with which we just worked.

***Exercise* #4:** Consider the division given by $\frac{3+2i}{4-5i}$.

(a) If division of complex numbers is consistent with division of real number then

$$\frac{4+5i}{4+5i}=$$

(b) Write the original quotient in simplest $a+bi$ form by multiplying it by the fraction in (a).

***Exercise* #5:** Find each of the following quotients in simplest $a+bi$ form.

(a) $\frac{2-4i}{5-3i}$

(b) $\frac{22+3i}{4-10i}$

(c) $\frac{10-5i}{3+4i}$

***Exercise* #6:** Which of the following represents the reciprocal of $3+9i$?

(1) $\frac{1}{3}+\frac{1}{9}i$

(2) $\frac{1}{9}-\frac{1}{3}i$

(3) $\frac{1}{30}-\frac{1}{10}i$

(4) $\frac{1}{10}+\frac{1}{3}i$

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MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS
ALGEBRA 2 WITH TRIGONOMETRY - HOMEWORK

SKILLS

1. Find each of the following products in simplest $a+bi$ form.

(a) $(5-2i)(-1+7i)$ (b) $(3+9i)(2+4i)$ (c) $(-4-i)(-2+6i)$

2. If the product of $(2+7i)$ and $(5+3i)$ were plotted, in which of the following quadrants would it lie?

(1) I (3) III

(2) II (4) IV

____

3. Multiply each of the following conjugate pairs by using the fact that $(a+bi)(a-bi)=a^2+b^2$.

(a) $(5-7i)(5+7i)$ (b) $(10+i)(10-i)$ (c) $(-3+8i)(-3-8i)$

4. The product of $(-8+2i)$ and its conjugate is equal to

(1) $64+4i$ (3) 68

(2) 60 (4) $60-4i$

____

5. The complex computation $(6+2i)(6-2i)-(3-4i)(3+4i)$ can be simplified to

(1) 15 (3) -10

(2) 39 (4) -35

____

6. Write each of the following quotients in simplest $a+bi$ form.

(a) $\frac{5+10i}{6-2i}$ (b) $\frac{11+2i}{3-4i}$ (c) $\frac{3+4i}{9-3i}$

(d) $\frac{2-4i}{2+4i}$ (e) $\frac{9-17i}{6+2i}$ (f) $\frac{26-2i}{12+3i}$

7. Find the reciprocal (multiplicative inverse) of each of the following complex numbers. Express your answers in simplest $a+bi$ form.

(a) $4-2i$ (b) $1+i$ (c) $\frac{3+4i}{5}$

8. Which of the following is equivalent to $\frac{7-i}{3+i}$?

(1) $2-i$ (3) $\frac{7}{3}+i$

(2) $\frac{11}{5}-i$ (4) $\frac{7}{3}-i$

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SOLVING QUADRATIC EQUATIONS WITH COMPLEX SOLUTIONS
ALGEBRA 2 WITH TRIGONOMETRY

As we saw in the last unit, the roots or zeros of any quadratic equation can be found using the quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Since this formula contains a square root, it is fair to investigate solutions to quadratic equations now when the quantity b^2-4ac, known as the **discriminant**, is negative.

***Exercise* #1:** Use the quadratic formula to find all solutions to the following equation. Express your answers in simplest $a+bi$ form. Check your answers by using the **STORE** feature on your calculator.

$$x^2-4x+29=0$$

As long as our solutions can include complex numbers, then any quadratic equation can be solved for two roots. We have actually seen quadratics already that have had complex roots in Lesson #1 of this unit.

***Exercise* #2:** Solve each of the following quadratic equations. Express your answers in simplest $a+bi$ form. Check your answers by using the **STORE** feature on your calculator.

(a) $x^2-5x+30=7x-10$

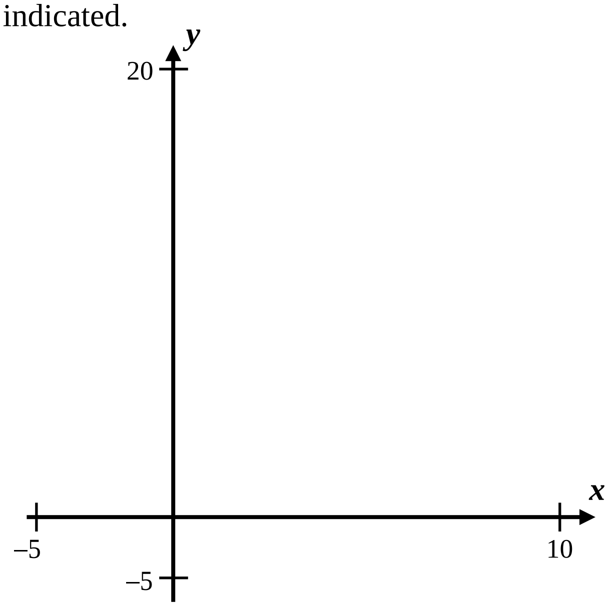
(b) $x^2+16x+15=10x+4$

There is an interesting connection between the x-intercepts of a parabola and complex roots with non-zero imaginary parts. This will be explored in the next lesson in more detail. But, the next exercise illustrates an important concept.

***Exercise* #3:** Consider the parabola whose equation is $y = x^2 - 6x + 13$.

(a) Algebraically find the x-intercepts of this parabola. Express your answers in simplest $a + bi$ form.

(b) Using your calculator, sketch a graph of the parabola on the axes below. Use the window indicated.



(c) From your answers to (a) and (b), what can be said about parabolas whose zeros are complex roots with non-zero imaginary parts?

***Exercise* #4:** Use the discriminant of each of the following quadratics to determine whether it has x-intercepts.

(a) $y = x^2 - 3x - 10$

(b) $y = x^2 + 6x + 10$

(c) $y = 2x^2 + 3x + 5$

***Exercise* #5:** Which of the following quadratic functions, when graphed, would not cross the x-axis?

(1) $y = 2x^2 + 5x - 3$

(2) $y = -x^2 - x + 6$

(3) $y = 4x^2 - 4x + 5$

(4) $y = 3x^2 - 13x + 4$

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SOLVING QUADRATIC EQUATIONS WITH COMPLEX SOLUTIONS
ALGEBRA 2 WITH TRIGONOMETRY - HOMEWORK

SKILLS

1. Solve each of the following quadratic equations. Express your solutions in simplest $a+bi$ form. Check.

(a) $x^2+4x+20=12x-5$

(b) $x^2=x-1$

(c) $2x^2-25x+27=-15x-10$

(d) $8x^2+36x+24=12x+5$

(e) $x^2+6x+15=8x-2$

(f) $4x^2+38x+50=10x-35$

2. Which of the following represents the solution set to the equation $x^2-2x+2=0$?

(1) $x=-1$ or 2 (3) $x=2\pm i$

(2) $x=1\pm 2i$ (4) $x=1\pm i$

______

3. The solutions to the equation $x^2+6x+11=0$ are

(1) $x=-3\pm i\sqrt{2}$ (3) $x=-6\pm i\sqrt{11}$

(2) $x=-3\pm 2i\sqrt{2}$ (4) $x=-6\pm 2i\sqrt{11}$

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4. Using the determinant, b^2-4ac, determine whether each of the following quadratics has real or imaginary zeros.

(a) $y=2x^2-7x+6$ (b) $y=3x^2+2x+1$ (c) $y=x^2-8x+14$

(d) $y=2x^2-12x+26$ (e) $y=-2x^2+6x-5$ (f) $y=4x^2-4x+1$

5. Which of the following quadratics, if graphed, would lie entirely above the x-axis. Try to use the discriminant to solve this problem and then graph to check.

(1) $y=2x^2+x-21$ (3) $y=x^2-4x+7$

(2) $y=x^2-x-6$ (4) $y=x^2-10x+16$

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REASONING

6. For what values of c will the quadratic $y=x^2+6x+c$ have no real zeros? Set up and solve an inequality for this problem.

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THE DISCRIMINANT OF A QUADRATIC
ALGEBRA 2 WITH TRIGONOMETRY

Since the roots of a quadratic can be found using $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, if a, b, and c are all rational numbers, the quantity under the square root, $b^2 - 4ac$, truly dictates what type of numbers the roots of a quadratic (and its x-intercepts) turn out to be. It reduces down to four cases which will be explored in *Exercise* #1.

***Exercise* #1:** For each of the following quadratics, calculate its discriminant, its roots, and state the number and nature (whether they are rational, irrational or imaginary) of the roots.

(a) Case I – The Discriminant is a Perfect Square - $x^2 + 3x - 10 = 0$.

$D = b^2 - 4ac =$ Roots: Number and Nature:

(b) Case II – The Discriminant is Not a Perfect Square - $x^2 - 6x + 7 = 0$.

$D = b^2 - 4ac =$ Roots: Number and Nature:

(c) Case III – The Discriminant is Equal to Zero - $x^2 - 10x + 25 = 0$.

$D = b^2 - 4ac =$ Roots: Number and Nature:

(d) Case IV – The Discriminant is Less than Zero - $x^2 - 8x + 20 = 0$

$D = b^2 - 4ac =$ Roots: Number and Nature:

In the last lesson, we explored Case IV extensively. In the case where the discriminant is negative, the roots of the quadratic are **imaginary** and it does not have *x*-intercepts (i.e. it does not cross the *x*-axis).

***Exercise* #2:** By using only the discriminant, determine the number and nature of the roots of each of the following quadratics.

(a) $2x^2+7x-4=0$

(b) $x^2-8x+25=0$

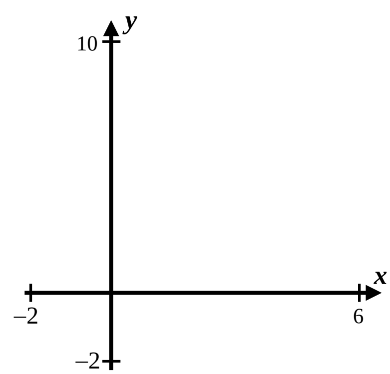
(c) $4x^2+4x+1=0$

(d) $x^2+6x+15=0$

(e) $4x^2-4x-7=0$

(f) $3x^2-7x+2=0$

***Exercise* #3:** Consider the quadratic function whose equation is $y=x^2-4x+4$. Determine the number of *x*-intercepts of this quadratic from the value of its discriminant. Then, sketch its graph on the axes given. We say that this parabola is **tangent** to the *x*-axis.



***Exercise* #4:** Which of the following parabolas has two unequal, rational *x*-intercepts?

(1) $y=x^2-2x-1$

(3) $y=x^2-8x+16$

(2) $y=x^2+2x-15$

(4) $y=x^2-3x+5$

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***Exercise* #5:** For what values of *a* will the parabola $y=ax^2+4x+2$ not cross the *x*-axis?

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The Discriminant of a Quadratic
Algebra 2 with Trigonometry - Homework

Skills

1. For each of the following quadratic equations, determine the number and the nature of the roots by first calculating the quadratic's discriminant.

(a) $2x^2+4x+5=0$ (b) $9x^2-12x+4=0$ (c) $4x^2-13x+3=0$

(d) $x^2+8x+11=0$ (e) $4x^2+4x-7=0$ (f) $36x^2-12x+1=0$

(g) $-3x^2+4x-8=0$ (h) $3x^2+8x+4=0$ (i) $x^2+8x+41=0$

2. The roots of $x^2+4x-7=0$ are

(1) unequal and rational (3) unequal and irrational

(2) unequal and imaginary (4) equal and rational

____

3. Which of the following quadratics has imaginary roots?

(1) $x^2+3x-5=0$ (3) $2x^2-3x+1=0$

(2) $x^2+6x+10=0$ (4) $x^2-7x+10=0$

____

4. Which of the following quadratic, when graphed, would touch the x-axis exactly once?

(1) $y=x^2-2x-3$ (3) $y=x^2+5x-2$

(2) $y=2x^2+3x+7$ (4) $y=x^2-12x+36$

____

5. If graphed, which of the following parabolas would lie entirely below the x-axis?

(1) $y = x^2 + 5x + 10$ (3) $y = -2x^2 + 6x - 5$

(2) $y = -2x^2 - 5x + 3$ (4) $y = x^2 + 6x + 9$

____

6. Which parabola below, when graphed, would cross the x-axis at two unequal irrational locations?

(1) $y = 2x^2 + 11x + 12$ (3) $y = 9x^2 - 6x + 1$

(2) $y = x^2 + 2x - 4$ (4) $y = 2x^2 + 4x + 9$

____

REASONING

7. Determine all values of a that will cause the parabola $y = ax^2 + 10x + 1$ to cross the x-axis at two distinct locations.

8. Consider the parabola whose equation is $y = x^2 - 4x$ and the line whose equation is $y = 2x + b$, where b is some unknown constant. Determine the value of b such that the line and the parabola will intersect at exactly one location. Then, sketch the system of equations on the axes below. Label their intersection point.