

UNIT #13

SEQUENCES AND SERIES

Lesson #1 – Sequences

Lesson #2 – Arithmetic and Geometric Sequences

Lesson #3 – Arithmetic Series

Lesson #4 – Geometric Series





Name: _____

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SEQUENCES ALGEBRA 2 WITH TRIGONOMETRY

Sequences, or an ordered list of numbers, are extremely important in mathematics, both theoretical and applied.

A **sequence** is formally defined as **a function that has as its domain the set** $\{1, 2, 3, \dots, n\}$.

Exercise #1: A sequence is defined by the equation $a(n) = 2n - 1$.

- (a) Find the first three terms of this sequence, denoted by a_1, a_2 , and a_3 .
(b) Evaluate the sum given by $\sum_{i=1}^5 a_i$.

Sequences can also be described by using **recursive definitions**. When a sequence is defined recursively, terms are found by operations on previous terms.

Exercise #2: A sequence is defined by the recursive formula: $a_{n+1} = a_n + 5$ with $a_1 = -2$.

- (a) Generate the first five terms of this sequence. Label each term with proper subscript notation.
(b) Determine the value of a_{20} . Hint – think about how many times you have added 5 to -2 .

Exercise #3: Determine a recursive definition for the sequence shown below. Be sure to include a starting value.

5, 10, 20, 40, 80, 160, ...

Exercise #4: For the recursively defined sequence $t_n = (t_{n-1})^2 + 2$ and $t_1 = 2$, the value of t_4 is

(1) 18

(3) 456

(2) 38

(4) 1446



Exercise #3: One of the most well-known sequences is the Fibonacci, which is defined recursively using two previous terms. Its definition is given below.

$$a_{n+1} = a_n + a_{n-1} \text{ and } a_1 = 1 \text{ and } a_2 = 1$$

Generate values for a_3 , a_4 , a_5 , and a_6 .

It is possible to find algebraic formulas for simple series, and this skill should be practiced.

Exercise #4: Find an algebraic formula $a(n)$, similar to that in *Exercise #1*, for each of the following sequences. Recall that the domain that you map from will be the set $\{1, 2, 3, \dots, n\}$.

(a) 4, 5, 6, 7, ...

(b) 2, 4, 8, 16, ...

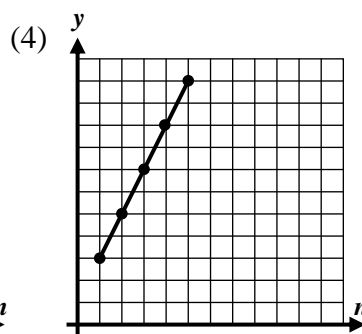
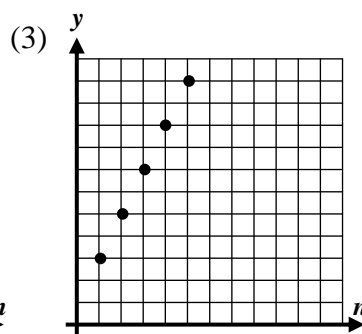
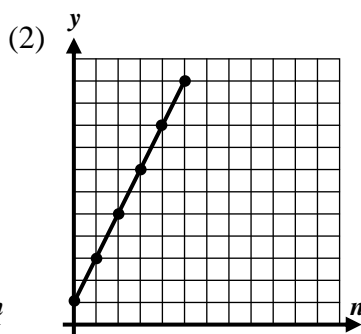
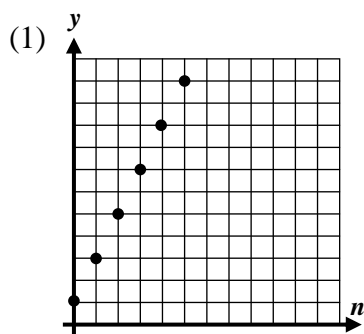
(c) $5, \frac{5}{2}, \frac{5}{3}, \frac{5}{4}, \dots$

(d) -1, 1, -1, 1, ...

(e) 10, 15, 20, 25, ...

(f) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots$

Exercise #5: Which of the following would represent the graph of the sequence $a_n = 2n + 1$?



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SEQUENCES
ALGEBRA 2 WITH TRIGONOMETRY - HOMEWORK

SKILLS

1. Given each of the following sequences defined by formulas, determine and label the first 4 terms. A variety of different notations is used below for practice purposes.

(a) $a(n) = 7n + 2$

(b) $a_n = n^2 - 5$

(c) $t(n) = \left(\frac{2}{3}\right)^n$

(d) $t_n = \frac{1}{n+1}$

2. Sequences below are defined recursively. Determine and label the **next** four terms of the sequence.

(a) $a_1 = 4$ and $a_n = a_{n-1} + 8$

(b) $t_n = t_{n-1} \cdot \frac{1}{2}$ and $t_1 = 24$

(c) $b_n = b_{n-1} + 2n$ with $b_1 = 5$

3. Given the sequence defined by $a_n = 3n - 1$, the value of $\sum_{i=1}^3 a_i =$

(1) 15

(3) 8

(2) 6

(4) 22

4. A recursive sequence is defined by $a_{n+1} = 2a_n - a_{n-1}$ with $a_1 = 0$ and $a_2 = 1$. Which of the following represents the value of a_5 ? _____

(1) 8

(3) 3

(2) -7

(4) 4 _____

5. Which of the following formulas would represent the sequence 10, 20, 40, 80, 160, ...

(1) $a_n = 10^n$

(3) $a_n = 5(2)^n$

(2) $a_n = 10(2)^n$

(4) $a_n = 2n + 10$ _____



6. For each of the following sequences, determine an algebraic formula, similar to *Exercise #4*, that defines the sequence.

(a) 5, 10, 15, 20, ...

(b) 3, 9, 27, 81, ...

(c) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

7. For each of the following sequences, state a recursive definition. Be sure to include a starting value or values.

(a) 8, 6, 4, 2, ...

(b) 2, 6, 18, 54, ...

(c) 2, -2, 2, -2, ...

APPLICATIONS

8. **Fibonacci's Rabbits** – Rabbits reproduce quickly. Let's say that we model an idealized population of rabbits using the fact that a pair of rabbits can reproduce when they are two months old and will produce one pair of rabbits of opposite gender each month thereafter. If we start with one pair of newborn rabbits in the first month, determine how many pairs of rabbits will exist in months 2, 3, 4, and 5.

REASONING

9. Consider a sequence defined similarly to the Fibonacci, but with a slight twist:

$$a_{n+1} = a_n - a_{n-1} \text{ with } a_1 = 2 \text{ and } a_2 = 5$$

Generate terms a_3, a_4, a_5, a_6, a_7 , and a_8 . Then, determine the value of a_{25} .



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ARITHMETIC AND GEOMETRIC SEQUENCES
ALGEBRA 2 WITH TRIGONOMETRY

There are many important types of sequences in mathematics, but the two most prominent are **arithmetic** and **geometric**. We will begin by defining the arithmetic sequence using a **recursive definition**.

ARITHMETIC SEQUENCE RECURSIVE DEFINITION

$$\text{Given } a_1, \text{ then } a_n = a_{n-1} + d$$

where d is called the **common difference** and can be positive or negative.

Exercise #1: Generate the next three terms of the given arithmetic sequences.

(a) $a_n = a_{n-1} + 6$ with $a_1 = 2$

(b) $a_1 = 8$ and $d = -4$

(c) $t_{n+1} = t_n + \frac{1}{2}$ and $t_1 = \frac{3}{2}$

Exercise #2: Given that $a_1 = -3 + 5i$ and $a_n = a_{n-1} + (6 - i)$, which of the following represents a_4 ?

(1) $15 + 2i$

(3) $9 + 3i$

(2) $21 + i$

(4) $3 + 4i$

It is important to be able to determine a general term of an arithmetic sequence based on the value of the index variable (the subscript). The next exercise walks you through the thinking process involved.

Exercise #3: Consider $a_n = a_{n-1} + 3$ with $a_1 = 5$.

(a) Determine the value of a_2 , a_3 , and a_4 .(b) How many times was 3 added to 5 in order to produce a_4 ?(c) Use your result from part (b) to quickly find the value of a_{50} .(d) Write a formula for the n^{th} term of an arithmetic sequence, a_n , based on the first term, a_1 , d and n .

Exercise #4: Given that $a_1 = 6$ and $a_4 = 18$ are members of an arithmetic sequence, determine the value of a_2 and a_{20} .

Geometric sequences are defined very similarly to arithmetic, but with a multiplicative constant instead of an additive one.

GEOMETRIC SEQUENCE RECURSIVE DEFINITION

Given a_1 , then $a_n = a_{n-1} \cdot r$

where r is called the **common ratio** and can be positive or negative and is often fractional.

Exercise #5: Generate the next three terms of the geometric sequences given below.

(a) $a_1 = 4$ and $r = 2$

(b) $a_n = a_{n-1} \cdot \frac{1}{3}$ with $a_1 = 9$

(c) $t_n = t_{n-1} \cdot \sqrt{2}$ with $t_1 = 3\sqrt{2}$

And, like arithmetic, we also need to be able to determine any given term of an geometric sequence based on the first value, the common ratio, and the index.

Exercise #6: Consider $a_1 = 2$ and $a_n = a_{n-1} \cdot 3$.

(a) Generate the value of a_4 .

(b) How many times did you need to multiply 2 by 3 in order to find a_4 .

(c) Determine the value of a_{10} .

(d) Write a formula for the n^{th} term of a geometric sequence, a_n , based on the first term, a_1 , r and n .



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ARITHMETIC AND GEOMETRIC SEQUENCES
ALGEBRA 2 WITH TRIGONOMETRY - HOMEWORK

SKILLS

1. Generate the next **three** terms of each arithmetic sequence shown below.

(a) $a_1 = -2$ and $d = 4$

(b) $a_n = a_{n-1} - 8$ with $a_1 = 10$

(c) $a_1 = 4 + 2i$ and $a_n = a_{n-1} - 2 + 3i$

2. In an arithmetic sequence $t_n = t_{n-1} + 7$. If $t_1 = -5$ determine the values of t_4 and t_{20} . Show the calculations that lead to your answers.

3. If $a_1 = 12$ and $a_n = a_{n-1} - 4$ then which of the following represents the value of a_{40} ?

(1) -148

(3) -144

(2) -140

(4) -172

4. If $c_1 = -5 + 8i$ and $d = 2 - 3i$ defines an arithmetic sequence of complex numbers then $c_{11} = ?$

(1) $15 - 22i$

(3) $22 - 33i$

(2) $17 - 25i$

(4) $-30 + 50i$

5. In an arithmetic sequence of numbers $a_1 = -4$ and $a_6 = 46$. Which of the following is the value of a_{12} ?

(1) 120

(3) 92

(2) 146

(4) 106

6. The first term of an arithmetic sequence whose common difference is 7 and whose 22nd term is given by $a_{22} = 143$ is which of the following?

(1) -25

(3) 7

(2) -4

(4) 28



7. Generate the next **three** terms of each geometric sequence defined below.

(a) $a_1 = -8$ with $r = -1$

(b) $a_n = a_{n-1} \cdot \frac{3}{2}$ and $a_1 = 16$

(c) $t_{n+1} = t_n \cdot -2$ and $t_1 = 5$

8. Given that $a_1 = 5$ and $a_2 = 15$ are the first two terms of a geometric sequence, determine the values of a_3 and a_{10} . Show the calculations that lead to your answers.

9. If the complex geometric sequence is defined by $c_1 = -7 + 4i$ and $r = i$, where $i = \sqrt{-1}$, then which of the following is the value of c_5 ?

(1) $7 - 4i$

(3) $-7 + 4i$

(2) $4 - 7i$

(4) $-4 + 7i$

10. In a geometric sequence, it is known that $a_1 = -1$ and $a_4 = 64$. The value of a_{10} is

(1) $-65,536$

(3) 512

(2) $262,144$

(4) -4096

APPLICATIONS

11. What would result in more money on the 31st day of the month: (1) Getting paid \$100 on the first day and an extra \$100 per day thereafter or (2) Getting paid 1 penny on the first day and having that amount double each day thereafter? Show calculations for both schemes to justify your answer.



ARITHMETIC SERIES
ALGEBRA 2 WITH TRIGONOMETRY

A **series** is simply the **sum of the terms of a sequence**. The fundamental definition/notion of a series is below.

THE DEFINITION OF A SERIES

If the set $\{a_1, a_2, a_3, \dots\}$ represent the elements of a sequence then the series, S_n , is defined by:

$$S_n = \sum_{i=1}^n a_i$$

In truth, you have already worked extensively with series in previous lessons almost anytime you evaluated a summation problem.

Exercise #1: Given the arithmetic sequence defined by $a_1 = -2$ and $a_n = a_{n-1} + 5$, then which of the following is

the value of $S_5 = \sum_{i=1}^5 a_i$?

(1) 32

(3) 25

(2) 40

(4) 27

The sums associated with arithmetic sequences, known as **arithmetic series**, have interesting properties, many applications and values that can be predicted with what is commonly known as **rainbow addition**.

Exercise #2: Consider the arithmetic sequence defined by $a_1 = 3$ and $a_n = a_{n-1} + 2$. The series, based on the first eight terms of this sequence, is shown below. Terms have been paired off as shown.

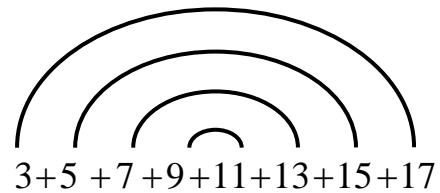
(a) What does each of the paired off sums equal?

(b) Why does it make sense that this sum is constant?

(c) How many of these pairs are there?

(d) Using your answers to (a) and (c) find the value of the sum using a multiplicative process.

(e) Generalize this now and create a formula for an arithmetic series sum based only on its first term, a_1 , its last term, a_n , and the number of terms, n .



SUM OF AN ARITHMETIC SERIES

Given an arithmetic series with n terms, $\{a_1, a_2, \dots, a_n\}$, then its sum is given by:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Exercise #3: Which of the following is the sum of the first 100 natural numbers?

- (1) 5,000 (3) 10,000
(2) 5,100 (4) 5,050
-

Exercise #4: Find the sum of each arithmetic series described or shown below.

(a) The sum of the sixteen terms given by:

$$-10 + -6 + -2 + \dots + 46 + 50.$$

(b) The first term is -8 , the common difference, d , is 6 and there are 20 terms

(c) The last term is $a_{12} = -29$ and the common difference, d , is -3 .

(d) The sum $5 + 8 + 11 + \dots + 77$.

Exercise #5: Kirk has set up a college savings account for his son, Maxwell. If Kirk deposits \$100 per month in an account, increasing the amount he deposits by \$10 per month each month, then how much will be in the account after 10 years?



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ARITHMETIC SERIES
ALGEBRA 2 WITH TRIGONOMETRY - HOMEWORK

SKILLS

1. Which of the following represents the sum of $3+10+\dots+87+94$ if the arithmetic series has 14 terms?
(1) 1,358 (3) 679
(2) 658 (4) 1,276

2. The sum of the first 50 natural numbers is
(1) 1,275 (3) 1,250
(2) 1,875 (4) 950

3. If the first and last terms of an arithmetic series are 5 and 27, respectively, and the series has a sum 192, then the number of terms in the series is
(1) 18 (3) 14
(2) 11 (4) 12

4. Find the sum of each arithmetic series described or shown below.
(a) The sum of the first 100 even, natural numbers.
(b) The sum of multiples of five from 10 to 75, inclusive.

(c) A series whose first two terms are -12 and -8 , respectively, and whose last term is 124.
(d) A series of 20 terms whose last term is equal to 97 and whose common difference is five.



5. For an arithmetic series that sums to 1,485, it is known that the first term equals 6 and the last term equals 93. *Algebraically* determine the number of terms summed in this series.

APPLICATIONS

6. Arlington High School recently installed a new black-box theatre for local productions. They only had room for 14 rows of seats, where the number of seats in each row constitutes an arithmetic sequence starting with eight seats and increasing by two seats per row thereafter. How many seats are in the new black-box theatre? Show the calculations that lead to your answer.
7. Simeon starts a retirement account where he will place \$50 into the account on the first month and increasing his deposit by \$5 per month each month after. If he saves this way for the next 20 years, how much will the account contain in principal?

REASONING

8. For an arithmetic series it is known that the first term is 2, the common difference is 5, and the sum is 87. Determine the number of terms in this series. Hint – you will need to solve a system of two equations.



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GEOMETRIC SERIES

ALGEBRA 2 WITH TRIGONOMETRY

Just as we can sum the terms of an arithmetic sequence to generate an arithmetic series, we can also sum the terms of a geometric sequence to generate a **geometric series**.

Exercise #1: Given a geometric series defined by the recursive formula $a_1 = 3$ and $a_n = a_{n-1} \cdot 2$, which of the following is the value of $S_5 = \sum_{i=1}^5 a_i$?

(1) 106

(3) 93

(2) 75

(4) 35

The sum of a finite number of geometric sequence terms is less obvious than that for an arithmetic series, but can be found nonetheless. The next exercise derives the formula for finding this sum.

Exercise #2: Recall that for a geometric sequence, the n th term is given by $a_n = a_1 \cdot r^{n-1}$. Thus, the general form of an arithmetic series is given below.

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}$$

(a) Write an expression below for the product of r and S_n .

$$r \cdot S_n =$$

(b) Find, in simplest form, the value of $S_n - r \cdot S_n$ in terms of a_1 , r , and n .

$$S_n - r \cdot S_n =$$

(c) Write both sides of the equation in (b) in their factored form.

(d) From the equation in part (c), find a formula for S_n in terms of a_1 , r , and n .

Exercise #3: Which of the following represents the sum of a geometric series with 8 terms whose first term is 3 and whose common ratio is 4?

(1) 32,756

(3) 42,560

(2) 28,765

(4) 65,535



SUM OF A FINITE GEOMETRIC SERIES

For a geometric series defined by its first term, a_1 , and its common ratio, r , the sum of n terms is given by:

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

Exercise #4: Find the value of the geometric series shown below. Show the calculations that lead to your final answer.

$$6 + 12 + 24 + \cdots + 768$$

Exercise #5: Maria places \$500 at the beginning of each year into an account that earns 5% interest compounded annually. Maria would like to determine how much money is in her account after she has made her \$500 deposit at the beginning of the 11th year.

- (a) Determine a formula for the amount, $A(t)$, that a given \$500 has grown to t -years after it was placed into this account.
- (b) Using this formula, determine the amount that \$500 deposited in years 1, 2, 10, and 11 have earned at the beginning of the 11th year.
- (c) Based on (b), write a geometric sum representing the amount of money in Maria's account after the 11th year deposit.
- (d) Evaluate the sum in (c) using the formula above.

Exercise #6: A person places 1 penny in a piggy bank on the first day of the month, 2 pennies on the second day, 4 pennies on the third, and so on. Will this person be a millionaire at the end of a 31 day month? Show the calculations that lead to your answer.



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GEOMETRIC SERIES
ALGEBRA 2 WITH TRIGONOMETRY - HOMEWORK

SKILLS

1. Find the sums of geometric series with the following properties:

(a) $a_1 = 6, r = 3$ and $n = 8$

(b) $a_1 = 20, r = \frac{1}{2}$, and $n = 6$

(c) $a_1 = -5, r = -2$, and $n = 10$

2. If the geometric series $54 + 36 + \dots + \frac{128}{27}$ has seven terms in its sum then the value of the sum is

(1) $\frac{4118}{27}$

(3) $\frac{1370}{9}$

(2) $\frac{1274}{3}$

(4) $\frac{8241}{54}$

3. A geometric series has a first term of 32 and a final term of $-\frac{1}{4}$ and a common ratio of $-\frac{1}{2}$. The value of this series is

(1) 19.75

(3) 22.5

(2) 16.25

(4) 21.25

4. Which of the following represents the value of $\sum_{i=0}^8 256 \left(\frac{3}{2}\right)^i$?

(1) 19,171

(3) 22,341

(2) 12,610

(4) 8,956

5. A geometric series whose first term is 3 and whose common ratio is 4 sums to 4095. The number of terms in this sum is

(1) 8

(3) 6

(2) 5

(4) 4

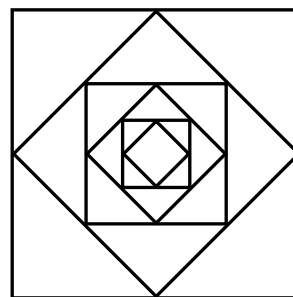


6. Find the sum of the geometric series shown below. Show the work that leads to your answer.

$$27 + 9 + 3 + \cdots + \frac{1}{729}$$

APPLICATIONS

7. In the picture shown at the right, the outer most square has an area of 16 square inches. All other squares are constructed by connecting the midpoints of the sides of the square it is inscribed within. Find the sum of the areas of all of the squares shown.



8. A college savings account is constructed so that \$1000 is placed the account on January 1st of each year with a guaranteed 3% yearly return in interest, applied at the end of each year to the balance in the account. If this is repeatedly done, how much money is in the account after the \$1000 is deposited at the beginning of the 19th year? Show the sum that leads to your answer as well as relevant calculations.
9. A ball is dropped from 16 feet above a hard surface. After each time it hits the surface, it rebounds to a height that is $\frac{3}{4}$ of its previous maximum height. What is the total vertical distance, to the nearest foot, the ball has traveled when it strikes the ground for the 10th time? Write out the first five terms of this sum to help visualize.

